Hence, for our particular model, we seek the solution of

$$h(Q,\tau) = A + BQ + CQ^2 = Q\,, \tag{34}$$

where

$$A = \int_0^\infty q_0(\tau)g(\tau)\,d\tau = \int_0^\infty \lambda[1-\beta\exp[-\alpha\tau]]\exp[-\lambda\tau]\,d\tau = 1 - \frac{\beta\lambda}{\alpha+\lambda}\,,$$

$$B = \int_0^\infty q_1(\tau)g(\tau)\,d\tau = \int_0^\infty \lambda(\beta-\gamma)\exp[-(\alpha+\lambda)\tau]\,d\tau = \frac{(\beta-\gamma)\lambda}{\alpha+\lambda}\,,$$

$$C = \int_0^\infty q_2(\tau)g(\tau)\,d\tau = \int_0^\infty \gamma\lambda\exp[-(\alpha+\lambda)\tau]\,d\tau = \frac{\gamma\lambda}{\alpha+\lambda}\,,$$

with $A+B+C=1$. Therefore, the probability that the cascade will eventually terminate is given by the smallest non-negative solution, less than unity, of

$$\gamma\lambda Q^2 - [\alpha - \lambda(1-(\beta+\gamma))]Q + \alpha + \lambda(1-\beta) = 0\,. \tag{35}$$

For values such that

$$[\alpha - \lambda(1-(\beta+\gamma))]^2 > 4\gamma\lambda(\alpha+\lambda(1-\beta))$$

two real roots, say ζ_1 and ζ_2, of (35) exist; and $Q = \min(\zeta_1, \zeta_2)$.

RIASSUNTO (*)

Nel presente lavoro si considera un modello semplice di una cascata di elettroni e fotoni in cui le probabilità di trasformazione sono funzioni dello spessore dell'assorbitore. Si ricavano la media e la varianza del numero degli elettroni nella cascata in funzione dello spessore dell'assorbitore e se ne discutono le proprietà. Si dà anche un'equazione che fornisce la probabilità che la cascata si arresti.

(*) *Traduzione a cura della Redazione.*

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Analytic Properties of Scattering Amplitudes as Functions of Momentum Transfer.

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Summary. — Scattering amplitudes are shown to have analytic properties as functions of momentum transfer. The partial wave expansions which define physical scattering amplitudes continue to converge for complex values of the scattering angle, and define uniquely the amplitudes appearing in the unphysical region of non-forward dispersion relations. The expansions converge for all values of momentum transfer for which dispersion relations have been proved.

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1. – Introduction.

The purpose of this note is to derive some properties of scattering amplitudes which follow from causality in relativistic quantum theory. It will be shown that a scattering amplitude has—for fixed energy—analytic properties as a function of scattering angle or momentum transfer. This consequence of causality is distinct from the existence of dispersion relations (1-4) which express analytic properties of a scattering amplitude as a function of energy for fixed momentum transfer. However, our results are of interest mainly in connection with dispersion relations for non-forward scattering. They imply that for all values of momentum transfer for which dispersion relations have

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(1) M. L. GOLDBERGER: *Phys. Rev.*, **99**, 979 (1955).

(2) K. SYMANZIK: *Phys. Rev.*, **100**, 743 (1957).

(3) N. BOGOLIUBOV, B. MEDVEDEV and M. POLIVANOV: *lecture notes*. Translated at the Institute for Advanced Study, (Princeton, 1957).

(4) H. J. BREMERMANN, R. OEHME and J. G. TAYLOR: *Phys. Rev.*, **109**, 2178 (1958). These papers contain numerous other references.

been established (3,4)—in π-N scattering this is the case for momentum tran $<\left(\frac{2}{3}\cdot(2m+\mu)/(2m-\mu)\right)^{\frac12}\cdot 4\mu$ —the so-called non-physical region is complet determined by the physical phase shifts via the partial wave expansion. Th possibility has been suggested previously (5).

We disregard spins and consider the scattering of a charged particle mass m (nucleon) with initial momentum p, final momentum p', and a neu particle of mass $\mu < 2m$ (meson) with initial momentum k, final momentum We assume in addition that these particles are not coupled to other parti of charge zero and mass $< 2\mu$ or of charge one and mass $< m+\mu$. Our res are valid also for the scattering of equal particles ($m=\mu$) under correspond restrictions on the mass spectrum.

As is well known, several equivalent expressions for the scattering ma in terms of Heisenberg operators may be given. With the notation

$$\langle p'k'\ \text{out}\,|\,pk\ \text{in}\rangle = \langle p'k'\ \text{in}\,|\,pk\ \text{in}\rangle + i(2\pi)\,\delta(p+k-p'-k')T$$

the amplitude T may be written as (6)

$$T = -\int \mathrm{d}^4x \exp\left[\frac{i(k+k')}{2}x\right]\langle p'|R'A\left(\frac{x}{2}\right)A\left(-\frac{x}{2}\right)|p\rangle\,, \tag{1}$$

or

$$T = -\int \mathrm{d}^4x \exp\left[\frac{i(k'-p')}{2}x\right]\langle 0|R'A\left(\frac{x}{2}\right)\psi\left(-\frac{x}{2}\right)|pk\ \text{in}\rangle\,. \tag{2}$$

$A(x)$ and $\psi(x)$ are the meson and nucleon field operators. R' denotes a ret commutator. For example,

$$R'A(x)\psi(y) = -i(\Box x-\mu^2)(\Box y - m^2)\,\theta(x-y)\,[A(x), \psi(y)]\,.$$

The state vectors refer to incoming or outgoing particles with definite mo as indicated.

Eq. (1) is used in the derivation of dispersion relations. Eq. (2) o other hand yields directly information about the scattering amplitud function of momentum transfer, since—in the center of mass system variable appears only in the exponential. However, more information tained in both cases by observing that from either (1) or (2) the fol

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(5) M. L. GOLDBERGER: *Proceedings of the Sixth Annual Rochester Co* (New York, 1956).

(6) H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN, *Nuovo Cimento*, **6**, 319

pression for the imaginary part of the amplitude may be derived,

$$) \quad \operatorname{Im} T = \pi\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2 \exp\left[\frac{i(k'-p')x_1}{2} - \frac{i(k-p)x_2}{2}\right]\cdot$$

$$\cdot\sum_\gamma \langle 0|R'A\left(\frac{x_1}{2}\right)\psi\left(-\frac{x_1}{2}\right)|p+k,\gamma\rangle\cdot\langle p+k,\gamma|R'A\left(\frac{x_2}{2}\right)\psi^+\left(-\frac{x_2}{2}\right)|0\rangle\,,$$

denotes a sum over all states with total four-momentum $p+k$. Eq. (3) is the energy shell) equivalent to the unitary requirement for the scattering plitude. It was first used by BOGOLIUBOV *et al.* (3). A simple proof of this tion is given in Sect. **3**.

The general method used to obtain explicit consequences of these expres s may be described as follows:

ach case the scattering amplitude—or its imaginary part—appears as the rier transform of a retarded commutator or of a sum over products of h commutators. Therefore it is simply related to the Fourier-transform he corresponding unretarded commutator which is in the case of (2) or (3) n by

$$F(q) = \int \mathrm{d}^4x \exp\,[iqx]\langle 0|\left[j\left(\frac{x}{2}\right), f\left(-\frac{x}{2}\right)\right]|p+k,\gamma\rangle\,,$$

e $j(x) = (\Box - \mu^2)A(x)$; $f(x) = (\Box - m^2)\psi(x)$. We know about $F(q)$:

it is the Fourier transform of a function that vanishes for space-like x;

(q) vanishes unless

$$\left\{\begin{array}{lll} \dfrac{p_0+k_0}{2}+q_0>0 & \text{and} & \left(\dfrac{p+k}{2}+q\right)^2 \geqslant m_1^2\,, \\ \text{or} & & \\ \dfrac{p_0+k_0}{2}-q_0\geqslant 0 & \text{and} & \left(\dfrac{p+k}{2}-q\right)^2 \geqslant m_2^2\,. \end{array}\right.$$

tter statement follows directly if a sum over intermediate states is ced in (4). m_1 and m_2 are the masses of the lowest intermediate states contribute to the two terms of the commutator. In the π-N case μ, $m_2 = m+\mu$.

DYSON (7) has solved the problem of finding a representation of all functi… satisfying these conditions. His result is: For $F(q)$ to satisfy (4a) and (… it is necessary and sufficient that it can be represented as

$$F(q) = \int \mathrm{d}^4u \int_0^\infty \mathrm{d}\varkappa^2 \varepsilon(q_0 - u_0)\delta[(q-u)^2 - \varkappa^2]\varphi(u, \varkappa^2)\,. \tag{5}$$

$\varphi(u, \varkappa^2)$ is arbitrary if the vectors $(p+k)/2+u$ and $(p+k)/2-u$ both li… the forward light-cone and

$$\varkappa \geqslant \mathrm{Max}\left\{0;\ m_1 - \sqrt{\left(\frac{p+k}{2}+u\right)^2};\ m_2 - \sqrt{\left(\frac{p+k}{2}-u\right)^2}\right\}.$$

φ vanishes outside this region. It depends, of course, also on the quant… numbers γ and on $p+k$. All our results will be based on application… Dyson's theorem.

For the Fourier transform of the retarded commutator which appears in … or (3) we have the relation $(q' = (q_0', \boldsymbol{q}))$

$$F_R(q) = -\frac{1}{2\pi}\int \frac{\mathrm{d}q_0' F(q')}{q_0' - q_0}\,; \qquad \mathrm{Im}\, q_0 \geqslant 0\,,$$

if F_R is sufficiently bounded.

Inserting (5) gives

$$F_R(q) = -\frac{1}{2\pi}\int \mathrm{d}^4u \int \frac{\mathrm{d}\varkappa^2\varphi(u, \varkappa^2)}{(q-u)^2 - \varkappa^2}\,. \tag{6}$$

In general we cannot expect F_R to be bounded enough for (6) to hold in… form. The necessary modification (8) does not alter the analytic prope… we are interested in. It is therefore sufficient to discuss (6). We shall… it both for Eq. (2) and Eq. (3).

2. – Momentum transfer properties of scattering amplitudes.

By inserting (6) into Eq. (2) we obtain

$$T = \frac{1}{2\pi}\int \frac{\mathrm{d}^4u\, \mathrm{d}\varkappa^2\varphi(u, \varkappa^2, p, k)}{((k'-p')/2-u)^2 - \varkappa^2}\,, \tag{7}$$

(7) F. J. DYSON: *Phys. Rev.*, **110**, 1460 (1958).
(8) R. JOST and H. LEHMANN: *Nuovo Cimento*, **5**, 1598 (1957), Eq. (4.5).

…an invariant function of the vectors u, p, k. The integration extends over … region given in (5). We choose the center of mass system to evaluate (7) … introduce the variables

$$\left|\begin{array}{l} W^2 = (p+k)^2; \qquad \Delta^2 = -\dfrac{(k-k')^2}{4}\,, \\ \text{or} \\ K^2 = \dfrac{[W^2-(m+\mu)^2][W^2-(m-\mu)^2]}{4W^2}; \qquad \cos\vartheta = 1 - \dfrac{2\Delta^2}{K^2}\,. \end{array}\right.$$

…pends then only on u^2, u_0, $\boldsymbol{u}\cdot\boldsymbol{k}$, $\varkappa^2$, W. It vanishes outside

$$\left\{\begin{array}{l} 0 \leqslant u \leqslant W/2; \quad -W/2 + u \leqslant u_0 \leqslant W/2 - u\,, \\ \varkappa \geqslant \mathrm{Max}\left\{0;\ m_1 - \sqrt{(W/2+u_0)^2 - u^2};\ m_2 - \sqrt{(W/2-u_0)^2-u^2}\right\}. \end{array}\right.$$

…ntroducing polar co-ordinates in $\boldsymbol{u}$-space, (7) becomes:

$$T(W, \cos\vartheta) = -\frac{1}{4\pi K}\int \mathrm{d}u_0 \int u\,\mathrm{d}u \int \mathrm{d}\varkappa^2 \int_0^{2\pi} \mathrm{d}\alpha \int_0^{\pi} \mathrm{d}\beta \cdot \frac{\varphi(u_0, u^2, \cos\alpha\sin\beta, \varkappa^2, W)}{\dfrac{K^2+u^2+\varkappa^2-(u_0+(m^2-\mu^2)/2W)^2}{2Ku\sin\beta} - \cos(\vartheta-\alpha)}\,,$$

$$T(W, \cos\vartheta) = \int_{x_0(W)}^{\infty} \mathrm{d}x \int_0^{2\pi} \mathrm{d}\alpha\, \frac{\varphi(x, \cos\alpha, W)}{x - \cos(\vartheta-\alpha)}\,,$$

$$\ldots\mathrm{s}\,\alpha, W) = -\frac{1}{4\pi K}\int \mathrm{d}u_0 \int u\,\mathrm{d}u \int \mathrm{d}\varkappa^2 \int_0^{\pi} \mathrm{d}\beta \cdot$$

$$\ldots\delta\left[x - \frac{K^2+u^2+\varkappa^2-(u_0+(m^2-\mu^2)/2W)^2}{2Ku\sin\beta}\right]\cdot\varphi(u_0, u^2, \cos\alpha\sin\beta, \varkappa^2, W)\,.$$

…wer limit $x_0(W)$ is determined by

$$x_0(W) = \mathrm{Min}\left\{\frac{K^2+u^2+\varkappa^2-(u_0+(m^2-\mu^2)/2W)^2}{2Ku}\right\},$$

if u_0, u, $\varkappa$ vary over the region (9). The minimum can be calculated in an elementary manner. The result is

$$x_0(W) = \left[1 + \frac{(m_1^2 - \mu^2)(m_2^2 - m^2)}{K^2[W^2 - (m_1 - m_2)^2]}\right]^{\frac{1}{2}} \tag{12}$$

In (11) the scattering angle $\cos\vartheta$ appears only in the denominator. We may therefore consider $\cos\vartheta$ as a complex variable and the scattering amplitude as an analytic function of $\cos\vartheta$. Moreover, this can be done separately for the real and imaginary parts of the amplitude.

Singularities of these functions can occur only if the denominator on the right hand side of (11) vanishes. That is for

$$\cos\vartheta = x\cdot\cos\alpha \pm i\sqrt{x^2 - 1}\,\sin\alpha\,.$$

We have therefore the following result:

(13) The real part and the imaginary part of the scattering amplitude are analytic functions of $\cos\vartheta$, regular inside an ellipse in the $\cos\vartheta$-plane with center at the origin and with axes x_0, $\sqrt{x_0^2 - 1}$.

We shall see presently—making use of Eq. (3)—that the imaginary part of the amplitude is regular in a larger domain, namely:

(14) $\operatorname{Im} T(W, \cos\vartheta)$ is regular in $\cos\vartheta$ inside an ellipse with center at the origin and with axes $2x_0^2 - 1$; $2x_0\cdot\sqrt{x_0^2 - 1}$. x_0 is given by (12).

Using $\cos\vartheta = 1 - (2\Delta^2/K^2)$ we can, of course, re-express (13) and (14) as analytic properties of the scattering amplitude as a function of momentum transfer.

These results (we defer the proof of (14)) lead—using well-known mathematical theorems (⁹)—to the following properties of the partial wave expansion of the scattering amplitude. Let

$$\begin{cases} T(W, \cos\vartheta) = \dfrac{1}{\pi^2}\dfrac{W}{K}\displaystyle\sum_{l=0}^{\infty}(2l+1)C_l(W)P_l(\cos\vartheta)\,, \\ \text{with}\quad C_l(W) = \dfrac{\pi^2}{2}\dfrac{K}{W}\displaystyle\int_{-1}^{1}\mathrm{d}\cos\vartheta\, T(W, \cos\vartheta)P_l(\cos\vartheta)\,. \end{cases} \tag{15}$$

(⁹) E. T. WHITTAKER and G. N. WATSON. *A course of modern analysis*, 4th Ed., (Cambridge 1940), p. 322; G. SZEGÖ. *Orthogonal Polynomials*, (New York, 1939), p. 238.

The Legendre series converges inside the domain of regularity of the represented functions; *i.e.* for $\cos\vartheta$ inside the ellipses (13) or (14) for $\operatorname{Re} T$ or $\operatorname{Im} T$ respectively. Also

$$\overline{\lim_{l\to\infty}}\,|\operatorname{Re} C_l(W)|^{1/l} \leqslant \frac{1}{x_0 + \sqrt{x_0^2 - 1}}\,, \tag{13a}$$

$$\overline{\lim_{l\to\infty}}\,|\operatorname{Im} C_l(W)|^{1/l} \leqslant \frac{1}{(x_0 + \sqrt{x_0^2 - 1})^2}\,. \tag{14a}$$

Taking into account the unitarity relation

$$\operatorname{Im} C_l(W) \geqslant [\operatorname{Re} C_l(W)]^2 + [\operatorname{Im} C_l(W)]^2\,, \tag{16}$$

we may note that (13a) is actually a consequence of (14a); *i.e.* if $\operatorname{Im} T(W, \cos\vartheta)$ is regular in the domain (14) it follows immediately that $\operatorname{Re} T(W, \cos\vartheta)$ is regular in (13).

We cannot conclude, of course, that the amplitude $T(W, \cos\vartheta)$ actually has singularities on the boundary of the domains (13) or (14). Using more physical information it may well be possible to improve these results.

To discuss the connection of the above statements with the non-physical region of dispersion relations, let us consider $\operatorname{Im} T(K^2, \Delta^2)$, the imaginary part of the amplitude, as a function of e.m. momentum and momentum transfer—the physical region is given by $K^2 > \Delta^2$. However, in the dispersion relation $\operatorname{Im} T(K^2, \Delta^2)$ is needed for all $K^2 > 0$. The expansion

$$\operatorname{Im} T(K^2, \Delta^2) = \frac{1}{\pi^2}\frac{W}{K}\sum_{l=0}^{\infty}(2l+1)\operatorname{Im} C_l(W)\,P_l\left(1 - \frac{2\Delta^2}{K^2}\right), \tag{15a}$$

defines a continuation of $\operatorname{Im} T(K^2, \Delta^2)$ into the non-physical region. The series converges if

$$\Delta^2 < K^2x_0^2 = K^2 + \frac{(m_1^2 - \mu^2)(m_2^2 - m^2)}{W^2 - (m_1 - m_2)^2}\,;$$

i.e. it converges for all $K^2 > 0$ provided

$$\Delta^2 < \operatorname{Min}\{K^2x_0^2\}\,.$$

This leads to the restriction

$$\Delta^2 < \frac{8}{3}\frac{2m + \mu}{2m - \mu}\cdot\mu^2 \approx 3\mu^2\,,$$

in the π-N case. ($\Delta^2 < 2\mu^2$ for equal particle scattering with $m_1 = m_2 = 2\mu$).

We have still to show that the continuation given by (15a) is indeed the correct definition of the non-physical region in the dispersion relation.

3. – Connection with dispersion relations.

To identify $\operatorname{Im} T(K^2, \Delta^2)$ as given by (15a) with the dispersion relation integrand it is convenient to consider (1) (which defines a function T for arbitrary real vectors k, k') not only on the energy shell ($k^2 = k'^2 = \mu^2$) but for the more general case $k^2 = k'^2 = \zeta$; keeping $p^2 = p'^2 = m^2$. T can then be considered as a function of

$$\omega = \frac{(k+k')(p+p')}{2\sqrt{(p+p')^2}} \qquad \zeta = k^2 = k'^2, \qquad \Delta^2 = -\frac{(p-p')^2}{4}.$$

We derive first Eq. (3).

(1) leads directly to

$$\text{(17)}\qquad \operatorname{Im} T(w, \zeta, \Delta^2) = \frac{1}{2}\int \mathrm{d}^4x \exp\left[i\left(\frac{k+k'}{2}\right)x\right]\langle p'|\left[j\left(\frac{x}{2}\right), j\left(-\frac{x}{2}\right)\right]|p\rangle = \\ = \frac{1}{2}\left\{M(\omega, \zeta, \Delta^2) - M(-\omega, \zeta, \Delta^2)\right\},$$

with

$$\text{(18)}\qquad M(\omega, \zeta, \Delta^2) = \int \mathrm{d}^4x \exp\left[i\left(\frac{k+k'}{2}\right)x\right]\langle p'|j\left(\frac{x}{2}\right), j\left(-\frac{x}{2}\right)|p\rangle = \\ = (2\pi)^4 \sum_\gamma \langle p'|j(0)|p+k, \gamma\rangle\langle p+k, \gamma|j(0)|p\rangle.$$

Let $\varphi\,\mathrm{in}\,(p')$ denote the annihilation operator for an incoming nucleon with momentum p'. Then

$$\langle p'|j(0)|p+k, \gamma\rangle = \langle 0|\varphi\,\mathrm{in}\,(p')j(0)|p+k, \gamma\rangle = \langle 0|[\varphi\,\mathrm{in}\,(p'), j(0)]|p+k, \gamma\rangle$$
$$\text{if}\quad (p+k-p')^2 = k'^2 = \zeta < 9\mu^2$$

since $\langle 0|j(0)\varphi\,\mathrm{in}\,(p')|p+k, \gamma\rangle = 0$ in this case. With the relation ([5])

$$[\varphi\,\mathrm{in}\,(p'), j(0)] = \frac{1}{(2\pi)^2}\int \mathrm{d}^4x \exp[ip'x]R'A(0)\psi(x),$$

and an analogous treatment of the second factor in (18) we have

$$\text{(19)}\quad M(\omega, \zeta, \Delta^2) = 2\pi\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2 \exp[ip'x_1 - ipx_2]\sum_\gamma \langle 0|R'A(0)\psi(x_1)|p+k, \gamma\rangle\cdot \\ \cdot\langle p+k, \gamma|R'A(0)\psi^+(x_2)|0\rangle = 2\pi\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2 \exp\left[i\left(\frac{k'-p'}{2}\right)x_1 - i\left(\frac{k-p}{2}\right)x_2\right]\cdot \\ \cdot\sum_\gamma \langle 0|R'A\left(\frac{x_1}{2}\right)\psi\left(-\frac{x_1}{2}\right)|p+k, \gamma\rangle\cdot\langle p+k, \gamma|R'A\left(\frac{x_2}{2}\right)\psi^+\left(-\frac{x_2}{2}\right)|0\rangle.$$

We note that the imaginary part of the physical scattering amplitude is given by

$$\text{(20)}\qquad \operatorname{Im} T = \tfrac{1}{2}M(\omega, \mu^2, \Delta^2).$$

The term $M(-\omega, \mu^2, \Delta^2)$ does not contribute since it vanishes for $\omega \geqslant \sqrt{\Delta^2+\mu^2}$. Hence (19) is on the energy shell equivalent to (3).

To find analytic properties of M as given by (19) we use again the integral representation (6). This leads to

$$\text{(21)}\quad M(\omega, \zeta, \Delta^2) = \frac{1}{2\pi}\int\frac{\mathrm{d}^4u_1\,\mathrm{d}\varkappa_1^2\,\mathrm{d}^4u_2\,\mathrm{d}\varkappa_2^2\,\Phi(u_1, \varkappa_1, u_2, \varkappa_2, p+k)}{[((k'-p')/2-u_1)^2-\varkappa_1^2][((k-p)/2-u_2)^2-\varkappa_2^2]},$$

where

$$\Phi(u_1, \varkappa_1, u_2, \varkappa_2, p+k) = \sum_\gamma \varphi_\gamma(u_1, \varkappa_1, p+k)\varphi_\gamma^*(u_2, \varkappa_2, p+k)$$

is built up from the weight functions φ_γ corresponding to the individual terms on the right hand side of (19). Φ is a real, invariant function which satisfies the support conditions (6) in each pair of variables u, $\varkappa$ separately. We choose the center of mass system and replace ω by

$$W^2 = 2\omega\sqrt{\Delta^2+m^2} + 2\Delta^2 + m^2 + \zeta.$$

Then

$$\text{(22)}\qquad M(\omega, \zeta, \Delta^2) = \frac{1}{2\pi}\cdot \\ \cdot\int\frac{\mathrm{d}^4u_1\,\mathrm{d}\varkappa_1^2\,\mathrm{d}^4u_2\,\mathrm{d}\varkappa_2^2\,\Phi(u_{10}, u_1^2, \varkappa_1^2, u_{20}, u_2^2, \varkappa_2^2, \mathbf{u}_1\mathbf{u}_2/u_1u_2, W)}{[((m^2-\zeta)/2W+u_{10})^2-\varkappa_1^2-(\mathbf{k}'-\mathbf{u}_1)^2][((m^2-\zeta)/2W+u_{20})^2-\varkappa_2^2-(\mathbf{k}-\mathbf{u}_2)^2]}.$$

With polar co-ordinates

$$\text{(23)}\qquad M(W, \zeta, \Delta^2)\,\frac{1}{8\pi K^2(\zeta)}\int \mathrm{d}u_{0}\,u_i\,\mathrm{d}u_i\,\mathrm{d}\varkappa_i^2\int_0^{2\pi}\mathrm{d}\alpha\int_0^{\pi}\mathrm{d}\beta_1\int_0^{\pi}\mathrm{d}\beta_2\int_0^{2\pi}\mathrm{d}\chi\cdot \\ \cdot\frac{\Phi(u_{0}, u_i^2, \varkappa_i^2, \cos\alpha\sin\beta_1\sin\beta_2+\cos\beta_1\cos\beta_2, W)}{[x_1(\zeta)-\cos(\vartheta-\chi)][x_2(\zeta)-\cos(\chi-\alpha)]},$$

$$x_i(\zeta) = \frac{K^2(\zeta) + u_i^2 + \varkappa_i^2 - ((m^2 - \zeta)/2W + u_{i0})^2}{2K(\zeta)u_i \sin \beta_i};$$

$$K^2(\zeta) = \frac{(W^2 + m^2 - \zeta)^2 - 4m^2W^2}{4W^2}.$$

We note that

$$\int_0^{2\pi} \mathrm{d}\chi \frac{1}{[x_1 - \cos(\vartheta - \chi)]} \cdot \frac{1}{[x_2 - \cos(\chi - \alpha)]} =$$

$$= 2\pi \frac{(x_1/(\sqrt{x_1^2 - 1})) + (x_2/(\sqrt{x_2^2 - 1})}{(x_1x_2 + \sqrt{x_1^2 - 1}\sqrt{x_2^2 - 1} - \cos(\vartheta - \alpha)}.$$

From (23) analytic properties in ζ and Δ^2 follow. On the energy shell, *i.e.* $\zeta = \mu^2$, we can introduce $y = x_1x_2 + \sqrt{x_1^2 - 1}\sqrt{x_2^2 - 1}$ as a new integration variable. Only integrations over y and α remain; the other integrations result only in a new weight function $\bar{\Phi}(y, \cos\alpha, W)$.

The minimum value of y is

$$y_0 = \mathrm{Min}\left\{x_1x_2 + \sqrt{x_1^2 - 1}\sqrt{x_2^2 - 1}\right\} = 2x_0^2 - 1.$$

Therefore

$$M(W^2, \Delta^2) = 2\,\mathrm{Im}\,T = \int_{2x_0^2 - 1}^{\infty} \mathrm{d}y \int_0^{2\pi} \mathrm{d}\alpha \frac{\bar{\Phi}(y, \cos\alpha, W)}{y - \cos(\vartheta - \alpha)}.$$

This proves the statement (14).

It can be seen now that the analytic continuation of $\mathrm{Im}\,T$ defined by (15*a*) yields the correct non-physical part of the dispersion relation integrand. In the proofs of these relations ([3,4]) it is shown first that—as a consequence of Eq. (1)—a dispersion relation in ω holds if ζ is taken real and $\zeta < -\Delta^2$. The absorptive part in this relation is $M(W, \zeta, \Delta^2)$ as given by (18) and (23). The dispersion relation for the physical value $\zeta = \mu^2$ is then obtained by analytic continuation in ζ, provided $M(W, \zeta, \Delta^2)$ is an analytic function of ζ regular for $\mathrm{Re}\,\zeta \leqslant \mu^2$ in a neighborhood of the real axis. It follows from (23) that this condition is satisfied if

$$\Delta^2 < \mathrm{Min}\,\{K^2 x_0^2\}.$$

This is also the condition for the convergence of the Legendre series. The absorptive part of the dispersion relation is then given by (23) with $\zeta = \mu^2$; *i.e.*, the non-physical region is obtained by analytic continuation in Δ^2 which can be carried out by the Legendre expression (15*a*).

The possibility of evaluating the non-physical region in this manner has been discussed earlier ([5]). While no proofs were given, it was believed on the basis of threshold arguments that such a procedure could be valid only if $\Delta^2 < \mu^2$, due to a branch point of the scattering amplitude as a function of Δ^2. We have shown that the expansion converges also for higher values of Δ^2; the limit being $\Delta^2 = 2\mu^2$ in the case of equal particle scattering. We believe that this is due to the fact that the real and imaginary part of the amplitude are separately analytic functions of Δ^2 and have different properties. For the dispersion relation only the imaginary part is needed and it has a larger domain of regularity. The mentioned branch point is likely to be present in the real part.

While we have no good reason to believe that our results are best possible, the expected appearance of a singularity in the real part gives us—via the unitarity relation (16)—an upper limit to the values of Δ^2 for which the Legendre expansion for the imaginary part might converge. In the case of equal particle scattering the expected branch point of the real part at $\Delta^2 = W^2/4$ leads to the limitation $\Delta^2 < 8\mu^2$ for the Legendre expansion of the imaginary part.

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RIASSUNTO (*)

Si dimostra che le ampiezze di scattering hanno proprietà analitiche come funzioni del trasferimento dei momenti. Gli sviluppi parziali delle funzioni d'onda che definiscono le ampiezze fisiche di scattering continuano a convergere per valori complessi dell'angolo di scattering e definiscono unicamente le ampiezze che compaiono nella regione non fisica delle relazioni di dispersione non in avanti. Gli sviluppi convergono per tutti i valori del trasferimento dei momenti per cui sono state dimostrate esatte le relazioni di dispersione.

(*) *Traduzione a cura della Redazione.*